The Cat in the Graph

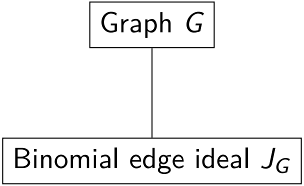
Armando Araujo-Gomez, **Isa Chou**, Adriana Garcia-Arias, Josiah Moltz

International REU in Commutative Algebra

April 7, 2026

Motivation

Binomial edge ideals are a class of ideals generated from graphs.



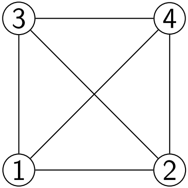
Binomial Edge Ideals

Definition

Given a graph $G = (V, E)$ with n vertices, the binomial edge ideal of G is the ideal J_G where

$$J_G = (x_i y_j - x_j y_i | \{i, j\} \in E(G))$$

in the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ over a field $\mathbb{K}$.



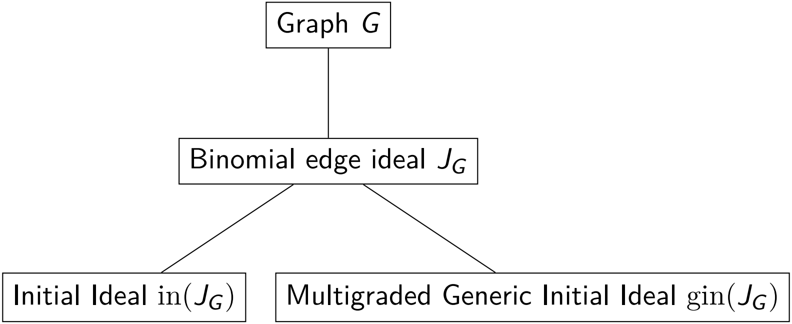
The graph on the left gives the generators of the following edge ideal:

$$J_{K_4} = (x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_1y_4 - x_4y_1,$$

$$x_2y_3 - x_3y_2, x_2y_4 - x_4y_2, x_3y_4 - x_4y_3).$$

Motivation

But binomials are hard. Square free monomial ideals are much easier to work with.



Initial Ideals

Definition

The **initial ideal** of J_G under a given monomial order $<$ is defined as:

$$\mathrm{in}_<(J_G) = (\mathrm{LM}(f) \mid f \in J_G \setminus \{0\})$$

where $\mathrm{LM}(f)$ is the leading monomial of f under $<$.

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Definition (Conca,De Negri, Gorla, 2021)

The **multigraded generic initial ideal** of J_G is

$$\mathrm{gin}(J_G) = (y_{a_1} \dots y_{a_v} x_i x_j \mid i, a_1, \dots, a_v, j \text{ is a path in } G).$$

Closed Graphs

If G is a closed graph, the initial and generic initial ideals of J_G look even nicer.

Definition

A graph G has a **closed labeling** if and only if $\mathrm{in}_<(J_G)$ is generated by the monomials $x_i y_j$ where $i < j$ and $\{i, j\}$ is an edge of G.

Definition

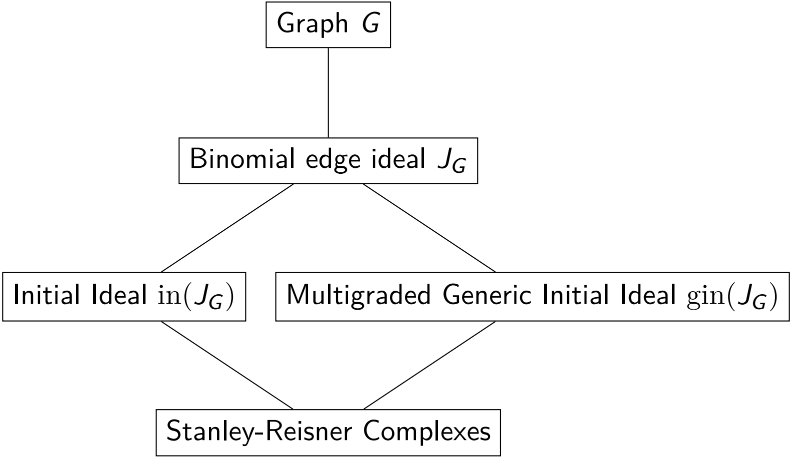
If there exists a closed labeling on G, then G is called a **closed graph.**

Combinatorial property of closed G

A graph is closed if and only if it is a proper interval graph. This also means if G has an induced 4-cycle (4-hole), it isn't closed.

Motivation

But algebra is hard. Combinatorics is easier.



Simplicial Complexes

Definition

A **simplicial complex** Δ on $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$ such that if $F \in \Delta$ and $F' \subseteq F$ then $F' \in \Delta$.

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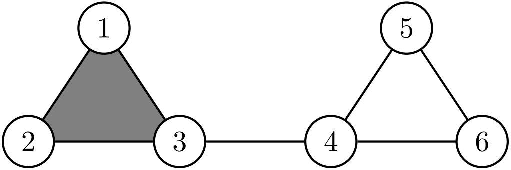
The elements $F \in \Delta$ are called *faces* and the maximal faces with respect to inclusion are called *facets*, the set of all the facets of Δ is denoted by $\mathcal{F}(\Delta)$.

For example, $\mathcal{F}(\Delta) = \{\{1,2,3\},\{3,4\},\{4,5\},\{5,6\},\{6,4\}\}$

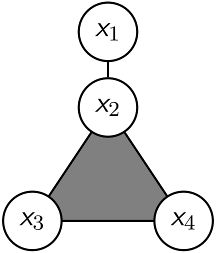


Stanley-Reisner Complex

Definition

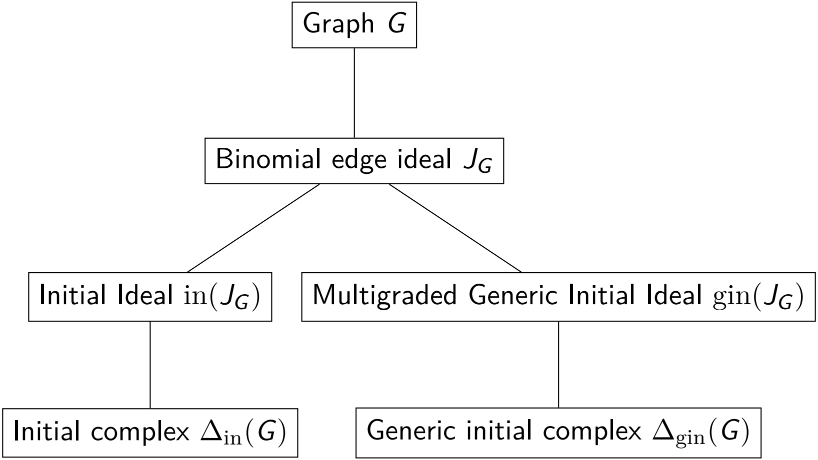
For square free monomial ideal $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$, we define the **Stanley-Reisner Complex** of I, denoted by Δ_I, to be the simplicial complex whose vertices are $\{x_1, \ldots, x_n\}$ and whose faces are $\sigma = \{x_{i_1}, \ldots, x_{i_t}\}$ such that $x_{i_1} \ldots x_{i_t} \notin I$.

If $I = (x_1x_3, x_1x_4) \subseteq \mathbb{K}[x_1, x_2, x_3, x_4]$ we have
$\mathcal{F}(\Delta_I) = \langle \{x_1, x_2\}, \{x_2, x_3, x_4\} \rangle$



Motivation

We consider simplicial complexes $\Delta_{\text{in}}(G)$ & $\Delta_{\text{gin}}(G)$ with Stanley-Reinser ideals $\text{in}(J_G)$ & $\text{gin}(J_G)$



Nice properties for a simplicial complex

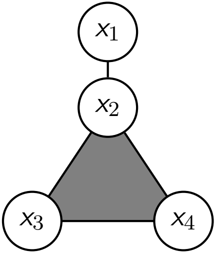
Nice properties for a simplicial complex

Pure

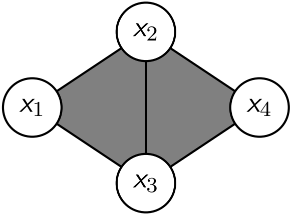
Vertex decomposable → Shellable

Definition

A simplicial complex is **pure** if all its facets have the same cardinality.



Non-pure simplicial complex



Pure simplicial complex

Nice properties for a simplicial complex

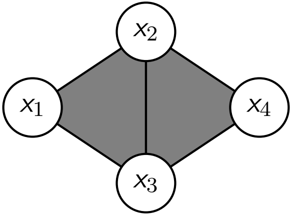
Definition

A simplicial complex Δ is **pure vertex decomposable** if Δ is pure and either

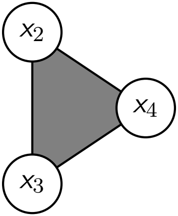
- $\Delta = \{\emptyset\}$ or Δ is a simplex, or
- there is a vertex $v \in \Delta$ such that $\mathrm{del}_\Delta(v) = \{F \in \Delta : F \cap \{v\} = \emptyset\}$ and $\mathrm{lk}_\Delta(v) = \{F \in \mathrm{del}_\Delta(v) : F \cup \{v\} \in \Delta\}$ are pure vertex decomposable.

Notion of vertex decomposable

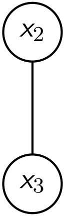
$$\Delta = \langle \{x_1, x_2, x_3\}, \{x_2, x_3, x_5\} \rangle$$



$\mathrm{del}_\Delta(x_1) = \{F \in \Delta : F \cap \{x_1\} = \emptyset\}$



$\mathrm{lk}_\Delta(x_1) = \{F \in \mathrm{del}_\Delta(x_1) : F \cup \{x_1\} \in \Delta\}$



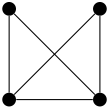
The problem

Problem

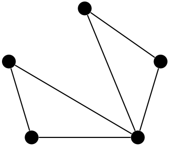
Let G be a closed and connected graph. For which G is $\Delta_{\text{in}}(G)$ pure, shellable and/or vertex decomposable? What about $\Delta_{\text{gin}}(G)$?

What did we find about G?

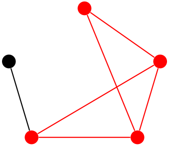
The cat graph!



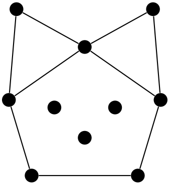
A cat graph



A cat-free graph



A graph with an induced cat



A cat-free graph (ironically)

Pre-existing Result (Modified)

Theorem 7.24 (Herzog, Hibi, Ohsugi, 2018)

Let G be a connected graph on $[n]$ which is closed with respect to a given labeling. Then the following conditions are equivalent:

1. (i) J_G is unmixed
2. (ii) J_G is Cohen-Macaulay
3. (iii) G is cat-free (ACGM, 2025)
4. (iv) The intersection of two maximal cliques of G has at most one vertex
5. (v) $\Delta_{\mathrm{in}}(G)$ & $\Delta_{\mathrm{gin}}(G)$ are pure (Bolognini, Macchia, Rinaldo, Strazzanti, 2025)

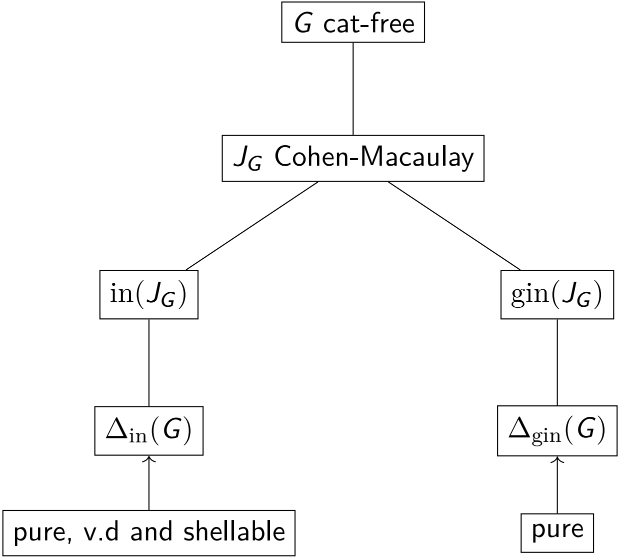
Initial Complex

Theorem (Van Tuyl, 2009)

Let G be a bipartite graph. The following are equivalent:

1. J_G is Cohen-Macaulay
2. $\Delta_{I(G)} = \Delta_{\mathrm{in}}(G)$ is pure shellable
3. $\Delta_{I(G)} = \Delta_{\mathrm{in}}(G)$ is pure vertex decomposable

If we consider G a closed, connected and cat-free,

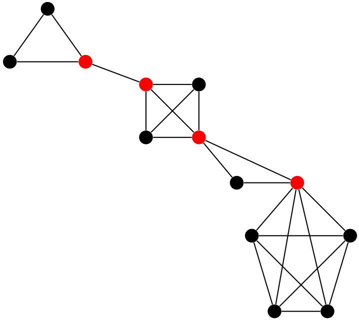


What about $\Delta_{\mathrm{gin}}(G)$?

Pivots

Definition (ACGM, 2025)

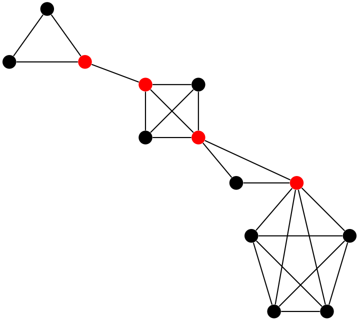
Given a closed and connected graph with maximal cliques $C_1, \ldots, C_r$ we define a **pivot** to be the intersection of two consecutive maximal cliques



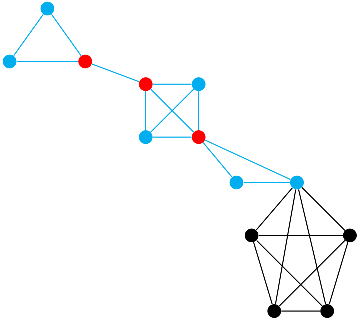
The big proof

Proposition (ACGM, 2025)

Let G be a cat-free graph, then $\Delta_{\text{gin}}(G)$ is pure vertex decomposable.

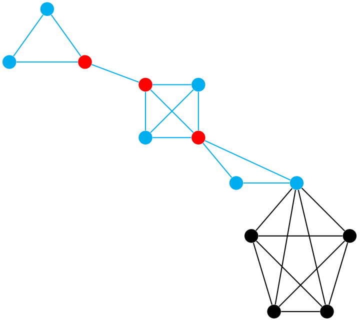


Inductive step

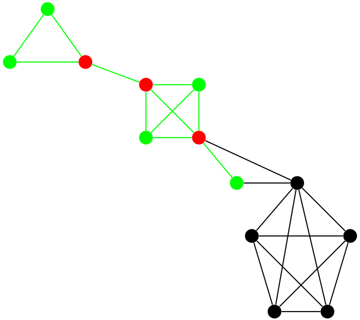


Induction Hypothesis

Induction hypothesis

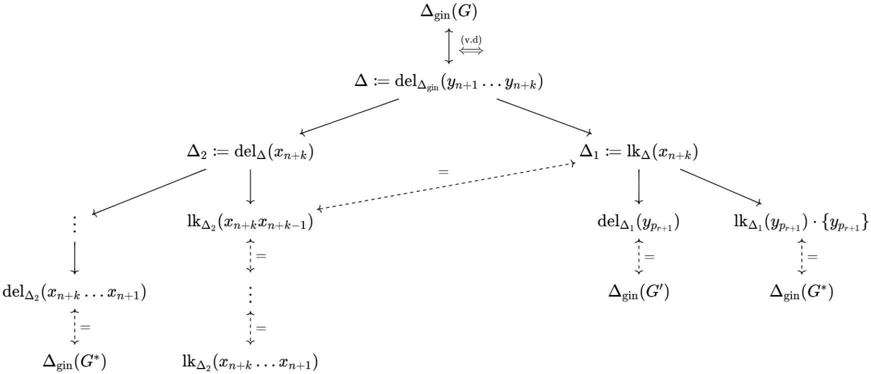


Graph G^*



Graph G'

Sketch of the proof



The big Theorem

Theorem (Cat Theorem)

Let G be a closed and connected graph. The following are equivalent:

1. G is cat-free (ACGM, 2025).
2. J_G is Cohen-Macaulay (Herzog, Hibi, Ohsugi, 2018).
3. $\Delta_{\mathrm{in}}(G)$ is pure. (Bolognini, Macchia, Rinaldo, Strazzanti, 2025)
4. $\Delta_{\mathrm{in}}(G)$ is pure shellable (Van Tuyl, 2009).
5. $\Delta_{\mathrm{in}}(G)$ is pure vertex-decomposable (Van Tuyl, 2009).
6. $\Delta_{\mathrm{gin}}(G)$ is pure (Bolognini, Macchia, Rinaldo, Strazzanti, 2025).
7. $\Delta_{\mathrm{gin}}(G)$ is pure shellable (ACGM, 2025).
8. $\Delta_{\mathrm{gin}}(G)$ is pure vertex decomposable (ACGM, 2025).

Conjecture

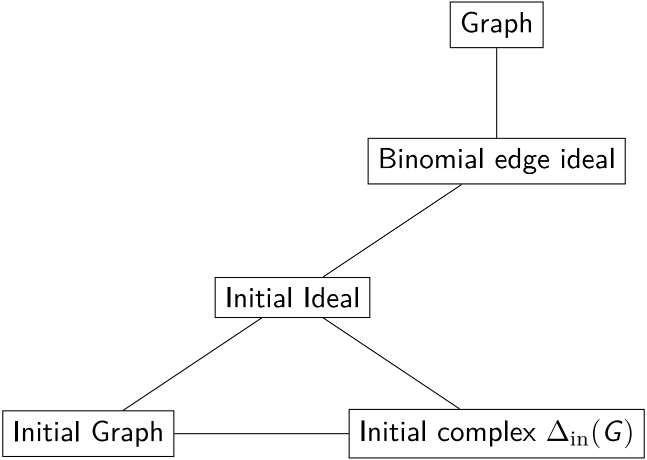
Conjecture

Let G be a closed and connected graph, then $\Delta_{\text{gin}}(G)$ is vertex decomposable even if G has a cat.

What about $\Delta_{\mathrm{in}}(G)$?

Motivation

If G has a closed labeling then studying $\Delta_{\text{in}}(G)$ can be reduced to studying the initial graph of G, $\text{in}(G)$.



$\mathrm{in}(G)$

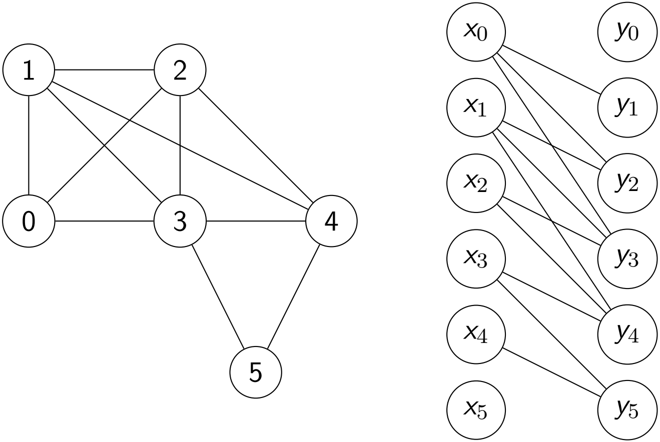
When G has a closed labeling, $\mathrm{in}(J_G) = (x_j y_j | i < j, \{i,j\} \in E(G)$.
$\mathrm{in}(G)$ is the interpretation of $\mathrm{in}(J_G)$ as an edge ideal.

Definition

The **initial graph** of closed G is $\mathrm{in}(G)$, the bipartite graph such that

1. $V(\mathrm{in}(G)) = \{x_i | i \in V(G)\} \cup \{y_i | i \in V(G)\}$
2. $E(\mathrm{in}(G)) = \{\{x_i, y_j\} | x_i y_j \in \mathrm{in}(J_G)\} = \{\{x_i, y_j\} | i < j, \{i,j\} \in E(G)\}$

What does taking an initial graph do?



Example of a closed labeling of a graph, and the corresponding initial graph

What are we doing here?

Problem

We would like to be able to recognize when a bipartite graph H is isomorphic to $\mathrm{in}(G)$ for some closed and connected graph G.

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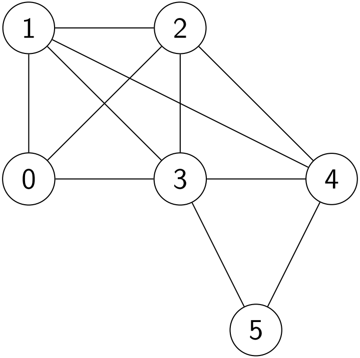
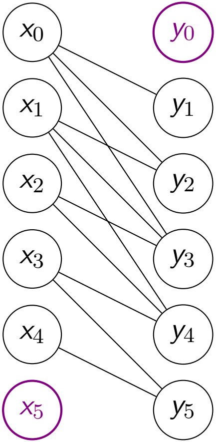
The Algorithm (ACGM, 2025)

We have a 2-part algorithm returning true if and only if $\exists G$ where $H \cong \mathrm{in}(G)$.
If it returns true, our algorithm returns an ordering $<$ such that $H^< = \mathrm{in}(G)$.

Degree 0 vertices

For all connected G with a closed labeling on $V(G) = \{0, \ldots, n-1\}$, x_{n-1} and y_0 are the only vertices of degree 0 in $\mathrm{in}(G)$.

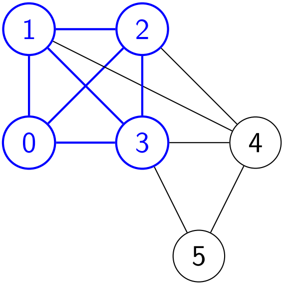


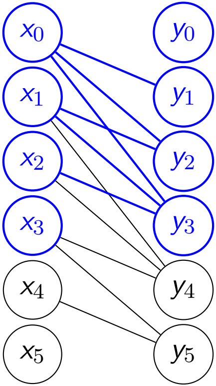


Stacking $\mathrm{in}(K_n)$

If G is connected and has a closed labeling, then the maximal cliques of G correspond to intervals of vertices. Thus G is a "stack" of K_ns (n may vary).

Thus, $\mathrm{in}(G)$ is a "stack" of $\mathrm{in}(K_n)$s.





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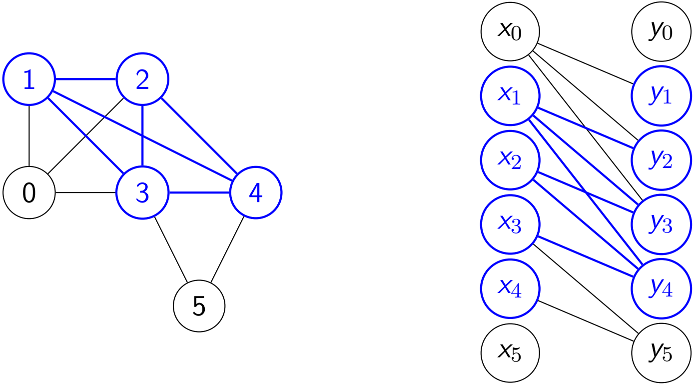

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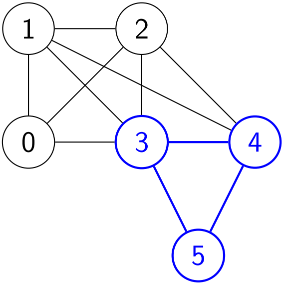
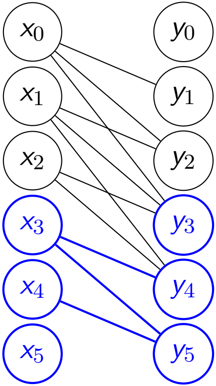




What does $\mathrm{in}(K_n)$ look like?

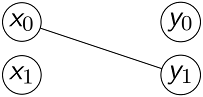


Figure: $\mathrm{in}(K_2)$

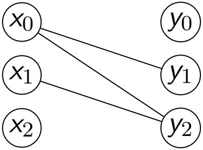


Figure: $\mathrm{in}(K_3)$

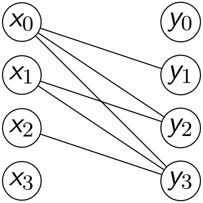


Figure: $\mathrm{in}(K_4)$

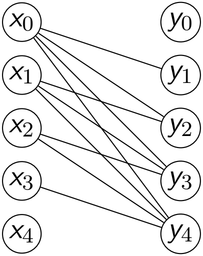


Figure: $\mathrm{in}(K_5)$

Combinatorial properties of $\mathrm{in}(K_n)$

1. There are two vertices of degree 1, x_{n-2} and y_1. Their neighbors are y_{n-1} and x_0.
2. Removing the vertices x_{n-2} and y_{n-1} from $\mathrm{in}(K_n)$ yields a graph isomorphic to $\mathrm{in}(K_{n-1})$.

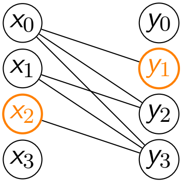


Figure: Degree 1 vertices

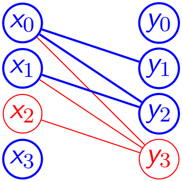
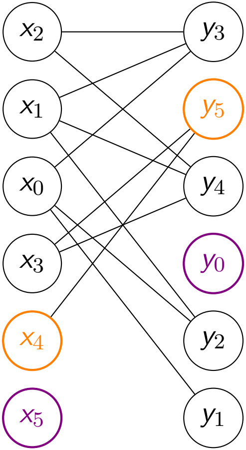


Figure: Removing x_2, y_3 from $\mathrm{in}(K_4)$ gives $\mathrm{in}(K_3)$

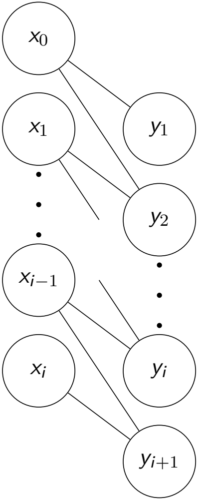
Algorithm Pt 1 overview

- x_5 and y_0 are vertices of degree 0.
- x_4 is a vertex of degree 1. It's neighbor is y_5.
- By removing x_4 and y_5 and repeatedly searching for degree 1 vertices, we find an ordering on this graph that makes it look like an initial graph.



What does this part actually do?

Iff the algorithm completes, it produces an ordering containing this zig-zag subgraph:

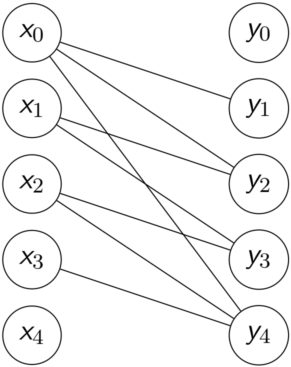


Lemma (ACGM, 2025)

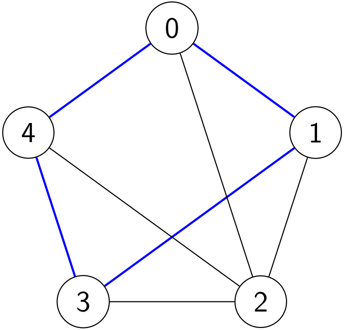
If a graph has this zig-zag structure, there are only two ways to draw it such that it's *down-right* ($\{x_i, y_j\} \in E(\text{in}(G))$ implies $i < j$).

An issue

However, we can exploit this structure to make a graph spuriously pass the algorithm.



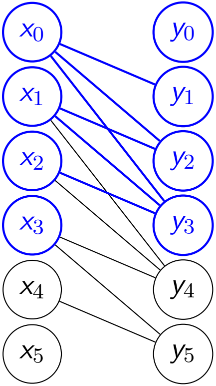
This graph contains the zig-zag, so it will pass the algorithm.



The graph it corresponds to is not closed. By the previous lemma, there is no other ordering.

But we have hope - recall again that because closed graphs have labelings such that maximal cliques are intervals, all $\mathrm{in}(G)$ consist of $\mathrm{in}(K_n)$s "stacked" on top of each other.

We thus add a condition that checks if these K_n do indeed exist:



Algorithm, part 2 (ACGM, 2025)

If $\deg(x_i) = d$ then the vertices $\{x_i, \ldots, x_{i+d}\} \cup \{y_i, \ldots, y_{i+d}\}$ must form an induced $\mathrm{in}(K_{d+1})$ for all $x_i \in X$.

The final theorem

Theorem (ACGM, 2025)

- If a graph H completes the algorithm and satisfies the previous condition, we indeed have $H \cong \mathrm{in}(G)$.
- Every $\mathrm{in}(G)$ will complete the algorithm and satisfy the previous condition.
- If $H \cong \mathrm{in}(G)$, the algorithm's ordering $<$ gives $H^< = \mathrm{in}(G)$.

Thank you!

Many thanks to the organizers of this session: Darren A. Narayan, Khang Tran, Christopher O'Neil, and Patricia Cahn.
Thank you so much to the REU organizers for their countless helpful discussions and for organizing the REU: Luis Núñez Betancourt[†], Eloísa Grifo*, Jack Jeffries*, Adam LaClair*, Yuriko Pitones[‡] Pedro Angel Ramírez Moreno[†], Alexandra Seceleanu*, Mark Walker *, Shahriyar Roshan Zamir*.

This research was funded by NSF RTG Grant DMS-2342256 *Commutative Algebra at Nebraska* and by SECIHTI grants CF-2023-G-33 *Redefiniendo fronteras entre el algebra conmutativa, la teoria de codigos y la teoria de singularidades* and Grant CBF 2023-2024-224: *Algebra conmutativa, singularidades y codigos*, as part of the the International REU in Commutative Algebra, held at CIMAT in Summer 2025, and organized by CIMAT and the University of Nebraska-Lincoln..

*University of Nebraska-Lincoln, [†] CIMAT, [‡] Universidad Autónoma Metropolitana