Pontryagin Duality and the Discrete Fourier Transform

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Abstract

This paper aims to survey Pontryagin duality and its connections to Fourier analysis. The Pontryagin dual was conceived by Lev Pontryagin in 1934, but was further generalized by van Kampen and is thus credited towards them both. It remains significant in algebraic topology and many more fields which seek to use Fourier analysis on objects other than $\mathbb{R}$ and $\mathbb{R}^n$. Possibly one of its most famous uses is in Tate's thesis (1950,) featuring a Fourier transform over the *Adelics.*

The Pontryagin dual generalizes the Fourier transform to any locally compact Hausdorff Abelian group. It is a deep construction that typically assumes knowledge of topology, measure theory and Banach algebras. We will attempt to describe it assuming minimal background in anything but Fourier analysis.

We begin by stating the essential background, from which we will formally construct the Pontryagin duality and describe its similarity to the Fourier Transform. Crucial results, such as the Pontryagin duality theorem, establish generalizations of well known properties of Fourier theory, and will be discussed. Many interesting applications arise from generalizing Fourier theory - a few of which will be discussed at the conclusion of this paper.

1 Characters and Locally Compact Abelian Groups

We'll begin our discussion with some necessary definitions and concepts.

Definition 1.1. Let G be an abelian group[1]. A *character* is a homomorphism[2]

$$\chi : G \to \mathbb{C}^*$$

where $\mathbb{C}^*$ denotes the multiplicative group of non-zero complex numbers.

Remark 1.1.1. If χ is a character on a finite abelian group G, $\chi(g)$ is a *root of unity*, since for each $g \in G$, $g^n = e$ for some $n \in \mathbb{N}$, and hence by the homomorphism properties of χ, we have $\chi(g)^n = \chi(g^n) = \chi(e) = 1$, where e is the identity element of G. Thus in the case of finite abelian groups, we can consider our characters to be maps $\chi : G \to S^1$, where S^1 is the multiplicative group of complex number of modulus 1. Furthermore, we have that the collection of n characters of G, $\{\chi_i\}_{i=0}^{n} \subseteq M_n$, where M_n is the multiplicative group of the n-th roots of unity.

Example 1.2. The *trivial* or *principal* character of G is the homomorphism $\mathbb{1}_G$ where $\mathbb{1}_G(g) = 1$ for all $g \in G$.

Proposition 1.3. Let G be an abelian group. Define the product of characters χ_1 and χ_2 by $(\chi_1\chi_2)(g) = (\chi_1)(g)(\chi_2)(g)$. We have that the set

$$\widehat{G} := \{\chi : G \to \mathbb{C}^* : \chi \text{ is a character}\}$$

forms an abelian group under point-wise multiplication. We call this group the *dual* group of G.

Remark 1.3.1. We make the distinction between the *Pontryagin dual* and *dual* group, even though we use the same notation to refer to them, $\widehat{G}$. The *Pontryagin dual* is merely a special case of the usual dual group.

Proof.

Let $\chi_1, \chi_2 \in \widehat{G}$ and consider the product $\chi_1\chi_2$.

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[1] An abelian group, also called a commutative group, is a group in which the group operation is commutative, i.e., $a * b = b * a$, where $*$ denotes the group operation.

[2] A homomorphism $\phi : G \to G'$ is a structure preserving map between two groups (or two rings, two vector spaces) that satisfies the property: $\phi(a * b) = \phi(a) * \phi(b)$.

Then for any $g, h \in G$, we have

$$\begin{aligned} \chi_1\chi_2(gh) &= \chi_1(gh)\chi_2(gh) \\ &= \chi_1(g)\chi_1(h)\chi_2(g)\chi_2(h) \\ &= \chi_1(g)\chi_2(g)\chi_1(h)\chi_2(h) \\ &= \chi_1\chi_2(g)\chi_1\chi_2(h) \end{aligned} \tag{1}$$

So, $\chi_1\chi_2 \in \widehat{G}$.
The associativity and commutativity of $\widehat{G}$ follow directly from the associativity and commutativity of the $\mathbb{C}^*$.
The principal character $\mathbb{1}_G$ is the identity of $\widehat{G}$ since we have

$$\begin{aligned} \mathbb{1}_G\chi(g) &= \mathbb{1}_G(g)\chi(g) \\ &= 1\chi(g) \\ &= \chi(g). \end{aligned} \tag{2}$$

Now for each $\chi \in \widehat{G}$, we have the inverse χ^{-1} of χ is given by $\chi^{-1}(g) = \overline{\chi(g)}$. Indeed, $\chi^{-1} \in \widehat{G}$ since we have

$$\begin{aligned} \chi^{-1}(gh) &= \overline{\chi(gh)} \\ &= \overline{\chi(g)\chi(h)} \\ &= \overline{\chi(g)}\,\overline{\chi(h)} \\ &= \chi^{-1}(g)\chi^{-1}(h), \end{aligned} \tag{3}$$

Therefore, $\widehat{G}$ is a group under point-wise multiplication. □

Having established the concept of characters of abelian groups, we will now turn our attention to *locally compact abelian* groups, which are essential in our discussion of Fourier analysis on groups.

Definition 1.4. A set G which has both group structure and topological space structure is a topological group if the maps

$$\begin{array}{ccc} G \to G & \text{and} & G \times G \to G \\ x \to x^{-1} & & (x, y) \to xy \end{array}$$

where $G \times G$ has the product topology, are continuous.

For instance, the following are topological groups as well as *Hausdorff* and *locally compact*:

1. $\mathbb{R}$ under the usual topology
2. The multiplicative group of $\mathbb{C}/1$ with the topology induced from the complex plane is called the circle group, and is denoted $\mathbb{T}$
3. Any group G endowed with the discrete topology produces a topological group.

Definition 1.5. A topological group G is called *locally compact* when it has a compact neighborhood of the identity; i.e, there exists a subset V of G containing the identity such that every point $x \in V$ has an *open* set around it. A topological group is additionally called *Hausdorff* iff the set $\{e\}$ is closed. (These are provably the same as the usual topological definitions)

Now, if the group is also Abelian, it is abbreviated as an *LCA-group* and will be written multiplicately from here on out.

2 Pontryagin Duality

Now, if G is any LCA-group, and $\mathbb{T}$ is the circle group as described above, consider the set of characters $\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$, i.e., the set of continuous homomorphisms $\widehat{G} : G \to \mathbb{T}$. It has abelian group structure under the operation given in *Proposition 1.3*, which follows naturally from the elements being homomorphisms of an LCA-group.

We assign it topological structure: if K is a compact subset of G and U is an open subset of $\mathbb{T}$, the sets $P(K, U) \subseteq \widehat{G}$ defined as

$$\left\{\chi : \chi \in \widehat{G} \text{ and } \chi(K) \subseteq U\right\}$$

are defined to be open and form a subbasis for the topology of $\widehat{G}$.

Referencing *Remark 1.3.1*, note that $\widehat{G}$ is an LCA-group.

Example 2.1. $\hat{\mathbb{R}} \cong \mathbb{R}$.

Proof.

Naturally for every $\xi \in \mathbb{R}$ we can define a character of $\mathbb{R}$:

$$\chi_d(x) = e^{2\pi i \xi x}, \quad x \in \mathbb{R}.$$

We claim these are the only characters of $\mathbb{R}$. Note $\chi\left(\frac{\varepsilon}{2}\right)^2 = \chi(\varepsilon) = e^{2\pi i \varepsilon \xi}$, so $\chi\left(\frac{\varepsilon}{2}\right) = \pm e^{2\pi i \frac{\varepsilon}{2}\xi}$, and $-e^{2\pi i \frac{\varepsilon}{2}\xi}$ does not have a positive real component. Repeating this, we have $\chi\left(\frac{\varepsilon}{2^n}\right) = e^{2\pi i \frac{\varepsilon}{2^n}\xi}$, and so for $k \in \mathbb{Z}$ we get

$$\chi\left(\frac{k}{2^n}\varepsilon\right) = \chi\left(\frac{\varepsilon}{2^n}\right)^k = e^{2\pi i \frac{k}{2^n}\varepsilon\xi}.$$

The set of rational numbers of the form $k/2^n, k \in \mathbb{Z}, n \in \mathbb{N}$, is dense in $\mathbb{R}$. Since χ is continuous, a small limit argument shows that $\chi(x) = e^{2\pi i x \xi}$ for everv $x \in \mathbb{R}$. Then, because $\chi_{\xi_1}\chi_{\xi_2} = \chi_{\xi_1 \xi_2}$, it is clear that $\hat{\mathbb{R}}$ is algebraically isomorphic to $\mathbb{R}$ under the isomorphism $\chi \to \chi_\xi$. It is additionally true that $\hat{\mathbb{R}}$ is indeed topologically isomorphic to $\mathbb{R}$ (meaning there is a map between the two which is both an algebraic isomorphism and a topological homeomorphism). ☐

For a more detailed proof, please see [2].

Example 2.2. $\hat{\mathbb{T}} \cong \mathbb{Z}$ and $\hat{\mathbb{Z}} \cong \mathbb{T}$.

Proof.

A rough outline is given. In $\mathbb{T}$, for any integer ξ we may define a character $\chi(e^{2\pi i x}) = e^{2\pi i \xi x}$. It is then possible to prove these are the only characters. Then, $\hat{\mathbb{T}}$ is seen to be algebraically isomorphic to $\mathbb{Z}$. Now, $\hat{\mathbb{T}}$ has the discrete topology and thus $\hat{\mathbb{T}}$ is topologically isomorphic to $\mathbb{Z}$.

In $\mathbb{Z}$, it is clear any character χ can be described by $\chi(1)$, since $\chi(n) = n\chi(1), n \in \mathbb{Z}$. Then for any $a \in \mathbb{T}$ we have a character χ_a where $\chi(1) = a$. Thus, $a \to \chi_a$ is an algebraic isomorphism between $\mathbb{T}$ and $\hat{\mathbb{Z}}$. It is then possible to show that $\hat{\mathbb{Z}}$ is topologically isomorphic to $\mathbb{T}$. ☐

Proposition 2.3 (Time-Frequency dictionary of Pontryagin Duality). Several LCA-groups and their dualities are given. Note $\mathbb{Q}_p$ and $\mathbb{Z}_p$ are p-adic completions of $\mathbb{Q}$ and $\mathbb{Z}$ respectively; K is a global field such as $\mathbb{Q}$, and $\mathbb{A}$ is its adele ring.

G	$\widehat{G}$
$\mathbb{R}$	$\mathbb{R}$
$\mathbb{Z}/n\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z}$
$\mathbb{Q}_p$	$\mathbb{Q}_p$
$\mathbb{A}$	$\mathbb{A}$
$\mathbb{Z}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{Z}$
$\mathbb{Z}_p$	$\mathbb{Q}_p/\mathbb{Z}_p$
K	$\mathbb{A}/K$
finite LCA	itself
$G \times H$	$\hat{G} \times \hat{H}$

We now state the most essential theorem in this area.

Theorem 2.4. (Pontryagin-van Kampen theorem) Let G be a locally compact Abelian group, and $\hat{\hat{G}}$ be the dual of its dual. For any fixed $x \in G$ and $g \in \hat{\hat{G}} : \hat{G} \to \mathbb{T}$ where $g(\chi) = \chi(x)$ for every $\chi \in \hat{G}$, the map $a : a(x) \to g$ provides a topological isomorphism $\hat{\hat{G}} \cong G$.

The proof is out of the scope of this paper. See [4], or [1] for a free resource.

3 The Haar Measure and Fourier Transform on Finite Abelian Groups

Definition 3.1 (Borel Sets)**.** Let G be a LCA group. A *Borel Set* in G is any set that can be formed from the open (or closed) sets of G through the operations of countable union, countable intersection, and relative complement. The collection of all Borel sets of a topological space G form a σ-algebra, and it is the smallest σ-algebra containing all open sets (or all closed sets).

Theorem 3.2 (Haar's Theorem)**.** There exists, up to a positive scaling factor, a unique countably additive, non-trivial measure μ on the Borel subsets of G that satisfies the following properties:

i.) The measure μ is left translation invariant:

$$\mu(gS) = \mu(S) \text{ for every } g \in G \text{ and all Borel subsets } S \subseteq G$$

ii.) The measure μ is finite on every compact set:

$$\mu(K) < \infty \text{ for all compact } K \subseteq G$$

iii.) The measure μ is outer regular on Borel sets $S \subseteq G$:

$$\mu(S) = \inf\{\mu\,(U)\,|S \subseteq U, U \text{ open}\}$$

iv.) The measure μ is inner regular on open sets $U \subseteq G$:

$$\mu(U) = \sup\{\mu\,(K)\,|K \subseteq U, K \text{ compact}\}$$

We call such a measure a *left Haar measure.*
We can similarly define a *right Haar measure.*

Example 3.3. We define a Haar measure μ on the circle group $\mathbb{T}$ by considering the function $g : [0, 2\pi] \to \mathbb{T}$ defined by $g(t) = \cos(t) + i \sin(t)$. Then we can define μ by

$$\mu(S) = \frac{1}{2\pi} l(g^{-1}(S)),$$

where l is the Lesbegue measure on $[0, 2\pi]$.

The Haar measure on G allows us to define the notion of integral for Borel functions defined on the group. We will first give a definition of L^p space with respect to our Haar measure μ, and with this, we are ready to tackle our main object of discussion.

Definition 3.4 (L^p space)**.** We define the L^p space associated with the Haar measure μ by

$$L^p_\mu\,(G) = \left\{f : G \to \mathbb{C} : \int_G |f(x)|^p d\mu(x) < \infty\right\}$$

Remark 3.4.1. Since any two Haar measures on G are equal upto a positive scaling factor, the L^p space is independent of the choice of Haar measure, and thus we can write it as $L^p(G)$.

Definition 3.5. If $f \in L^1(G)$, then the *Fourier transform* of f is the function $\widehat{f}$ defined on $\widehat{G}$ which is given by

$$\widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}d\mu(x)$$

where the integral is relative to the Haar measure μ.
If $g \in L^1(\widehat{G})$, then the *inverse Fourier transform* of g is given by

$$\check{g}(x) = \int_{\widehat{G}} g\left(\chi\right) \chi\left(x\right) d\widehat{\mu}\left(\chi\right)$$

where the integral is relative to the dual Haar measure $\widehat{\mu}$ of μ.

Example 3.6. Consider the simple function $f(k)$ defined on $\mathbb{Z}/n\mathbb{Z}$ where

$$f(k) = 1 \text{ for all } k \in \mathbb{Z}/n\mathbb{Z}.$$

This is the constant function on the group, and it will illustrate the basic computation well. The fourier transform $\widehat{f}$ at a point $j \in \mathbb{Z}/n\mathbb{Z}$ (Recall our dual group $\widehat{G}$ is $\mathbb{Z}/n\mathbb{Z}$ in this instance) is given by:

$$\begin{aligned} \widehat{f}(j) &= \int_{\mathbb{Z}/n\mathbb{Z}} f(k)e^{-2\pi ijk/n}d\mu(n) \\ &= \sum_{k=0}^{n-1} f(k)e^{-2\pi ijk/n} \\ &= \sum_{k=0}^{n-1} e^{-2\pi ijk/n} \\ &= \begin{cases} n & j \equiv 0 \bmod n \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

This summation is actually the sum of the nth roots of unity. The sum of all nth roots of unity is 0 for any j not divisible by n (i.e., $j \not\equiv 0 \bmod n$), and it is n when $j \equiv 0 \bmod n$ because each term of the sum is 1.

Note that when we integrate over discrete groups like $\mathbb{Z}/n\mathbb{Z}$, we assign each element of the group with a Haar measure of 1. So in essence, integrating over discrete groups amounts to summing over the group elements. This follows from the fact that we endow discrete groups G with the discrete topology.

We'll give some more useful properties of our Fourier theory over LCA groups.

Definition 3.7 (Convolution)**.** Let $f, g \in L^1(G)$. The convolution of f and g is defined as

$$(f * g)(x) = \int_G f(x-y)g(y)d\mu(y).$$

Theorem 3.8 (Convolution to Multiplication)**.** Given $f, g \in L^1(G)$, the Fourier transform takes convolution to multiplication, i.e.,

$$\widehat{f * g}(\chi) = \widehat{f}(\chi) \cdot \widehat{g}(\chi).$$

Proof.

We have that the Fourier transform of the convolution $f * g$ is given by:

$$\begin{aligned}
\widehat{f * g}(\chi) &= \int_G (f * g)(x)\overline{\chi(x)}d\mu(x) \\
&= \int_G \left(\int_G f(x-y)g(y)d\mu(y)\right)\overline{\chi(x)}d\mu(x) \\
&= \int_G g(y) \int_G f(x-y)\overline{\chi(x)}d\mu(x)d\mu(y) \\
&= \int_G g(y)\overline{\chi(y)} \int_G f(x-y)\overline{\chi(x-y)}d\mu(x)d\mu(y) \quad \left(\overline{\chi(x)} = \overline{\chi(xy^{-1})}\,\overline{\chi(y)}\right) \\
&= \int_G g(y)\overline{\chi(y)}\widehat{f}(\chi)d\mu(y) \quad \text{(By the translation invariance of } \mu) \\
&= \widehat{f}(\chi) \cdot \int_G g(y)\overline{\chi(y)}d\mu(y) \\
&= \widehat{f}(\chi) \cdot \widehat{g}(\chi)
\end{aligned}$$

□

Theorem 3.9 (Plancherel)**.** Given a Haar measure μ and the dual measure $\widehat{\mu}$, if $f : G \to \mathbb{C}$ is continuous with compact support, then $\widehat{f} \in L^2(G)$ and

$$\int_G |f(x)|^2 d\mu(x) = \int_{\widehat{G}} |\widehat{f}(\chi)|^2 d\widehat{\mu}(\chi).$$

4 Shannon Sampling Theorem on LCA-Group

One of the most relevant applications of Fourier analysis is the Shannon Sampling Formula, it is essential for digital signal processing as a way of reconstructing a one-dimensional signal from a set of samples. The sampling formula is only valid for functions $f \in \mathcal{S}(\mathbb{R})$ whose Fourier transform vanishes beyond the interval $[-\tau/2\pi, \tau/2\pi]$.

Theorem 4.1. (Shannon Sampling Formula). Suppose $f \in \mathcal{S}(\mathbb{R})$ such that $\widehat{f}$ is supported on the interval $[-\tau/2\pi, \tau/2\pi]$. Then

$$f(x) = \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{\tau}\right) sinc\left[\tau\left(x - \frac{n\pi}{\tau}\right)\right].$$

Where

$$sinc(x) = \begin{cases} \frac{sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Proof.

Given $f \in \mathcal{S}(\mathbb{R})$ such that its Fourier transform $\widehat{f}$ is supported on [-L/2,L/2] where $L = \tau/\pi$, we are going to view $\widehat{f}$ as an L-periodic function.

$$\widehat{f}(\xi) = \sum_{n\in\mathbb{Z}} a_L(n) e^{\frac{2\pi i n\xi}{L}},$$

where the L-Fourier coefficients are given by

$$\begin{aligned} a_L(n) &= \frac{1}{L}\int_{-\infty}^{\infty} \widehat{f} e^{\frac{-2\pi i n\xi}{L}} d\xi \;=\; \frac{1}{L}\int_{-\infty}^{\infty} \widehat{f} e^{2\pi i \frac{-n}{L}\xi} d\xi \\ &= \frac{1}{L}\check{\widehat{f}}\Big(\frac{-n}{L}\Big) \;=\; \frac{1}{L} f\Big(\frac{-n}{L}\Big). \end{aligned}$$

Now we substitute $a_L(n)$ into the original equation

$$\widehat{f}(\xi) = \sum_{n\in\mathbb{Z}} \frac{1}{L} f\Big(\frac{n}{L}\Big) e^{\frac{-2\pi i n\xi}{L}}.$$

From here we compute the inverse Fourier transform to find $f(x)$

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i x\xi} d\xi \\ &= \int_{-L/2}^{L/2} \sum_{n\in\mathbb{Z}} \frac{1}{L} f\left(\frac{n}{L}\right) e^{2\pi i\xi(x-\frac{n}{L})} d\xi \\ &= \sum_{n\in\mathbb{Z}} f\left(\frac{n}{L}\right) \frac{1}{L} \int_{-L/2}^{L/2} e^{2\pi i\xi(x-\frac{n}{L})} d\xi. \end{aligned}$$

Now we will calculate the integral bit of the equation

$$\begin{aligned} \frac{1}{L}\int_{-L/2}^{L/2} e^{2\pi i\xi(x-\frac{n}{L})} d\xi &= \frac{1}{L}\frac{e^{2\pi i\xi(x-\frac{n}{L})}}{2\pi i(x-\frac{n}{L})}\Bigg|_{\xi=\frac{-L}{2}}^{\xi=\frac{L}{2}} \\ &= \frac{1}{L}\left(\frac{e^{i\pi L(x-\frac{n}{L})} - e^{-i\pi L(x-\frac{n}{L})}}{2i}\right)\frac{1}{\pi(x-\frac{n}{L})} \\ &= \frac{sin(\pi L(x-\frac{n}{L})}{\pi L(x-\frac{n}{L})} \\ &= sinc(\pi L(x-\frac{n}{L})). \end{aligned}$$

Putting this into our equation for $f(x)$ we get

$$f(x) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{L}\right) sinc(\pi L(x-\frac{n}{L})).$$

Now if set our $L = \frac{\tau}{\pi}$ we get the Shannon Sampling Formula

$$f(x) = \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{\tau}\right) sinc\left[\tau\left(x-\frac{n\pi}{\tau}\right)\right].$$

□

However, in this section we will be discussing a Shannon Sampling Formula for an additive LCA-group: G.

In our sampling formula for $\mathbb{R}$ the sampling points are given by $n\pi/\tau \in \mathbb{Z}$, now we will redefine the sampling points for our LCA-group as $h \in H$, where H is a discrete subgroup of G. However, since this formula is to be general to an LCA we must also redefine *sinc* with another suitable function ϕ. ϕ

is defined as the inverse Fourier transform of the homomorphism $\mathbb{1}_A$, where A is the annihilator of H defined by $A = \{\chi \in \widehat{G} : \langle h, \chi \rangle = 1\}$, this is also equivalent to saying that A is isomorphic to G/H.

Theorem 4.2. (Sampling Theorem for LCA-group). Suppose $f \in L^2(G)$, H a discrete subgroup of G, and with Fourier transform $\widehat{f}$ supported on A the annihilator of H then

$$f(h) = \sum_{h \in H} f(h)\phi(g-h)$$

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